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Depathing maps for circulating state shifts

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Dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday.

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Abstract

We have described a decirculation process which marks perturbations of network structure that are necessary for nonlinear network dynamics to proceed from one circulating state (a limit cycle) to another stable state (a limit cycle or a fixed point). Armed with the decirculation process, a sort of decirculating maps and their structural properties have also been built, dedicated to showing that circulation breaking taking place in nonlinear network dynamics can collaborate harmoniously toward the completion of network structure that generates attractors (equilibrium states). Here we wish to extend the notion of decirculating maps to the notion of depathing maps. The extension allows us to reshape network structure not only on the occasion of circulating states but on the occasion of any required path states. This gives a crucial improvement in generating circulating state shifts more feasibly.

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1 Introduction

Pattern generation in complex biological systems may be understood by means of the concepts of nonlinear network dynamics [1, 2]. The modeled systems can be formed by large numbers of interacting units whose dynamical properties tend to emerge through the collective interactions of many units. The modeled systems generally reach one of possible multiple stable states (alternative stable states) [3–5], which have multistability governed by the control parameters assigned to evolutionary network structure. State shifts between multiple stable states can be induced by the decirculation process [6], which marks a quantified determinant of the reshape of network structure that is sufficient for shifts from one circulating state (a limit cycle) to another stable state (a limit cycle or a fixed point). The decirculation process is generally stated as follows: 'the occurrence of a loop of unit states in the modeled systems leads to a change in network connections, which feeds back to reinforce interacting units to tend to break the circulation of unit states in this loop.'

Armed with the decirculation process, a sort of decirculating maps and their structural properties are built in [7, 8], dedicated to showing that circulation breaking taking place in nonlinear network dynamics can collaborate harmoniously toward the completion of network structure that generates attractors (equilibrium states). Here we wish to extend the notion of decirculating maps to the notion of depathing maps. The extension allows us to reshape network structure not only on the occasion of circulating states but on the occasion of any required path states. Hence it can generate circulating state shifts more feasibly. It reveals the depathing process which is generally stated as follows: 'the occurrence of a path of unit states in the modeled systems leads to a change in network connections, which feeds back to reinforce interacting units to tend to break the flow of unit states in this path.' Operator construction for path breaking is also put in the section at the end, displaying the tendency toward path breaking aiming to control nonlinear network dynamics.

2 Depathing maps

Let $\{0,1\}^n$ denote the binary code consisting of all 01-strings of fixed length n. Denote by $\Omega = [x^0, x^1, \ldots, x^p]$ a path of states in $\{0,1\}^n$, meaning that $p > 1$, $x^0, x^1, \ldots, x^p \in \{0,1\}^n$, and $x^0 \neq x^i$ for some $i \in \{1,2,\ldots,p\}$. Specifically, we call that Ω is a loop if $x^0 = x^p$.

For every $i, j = 1,2,\ldots,n$, we assign an integer, denoted by $c_{ij}(\Omega)$, according to the rule

$$c_{ij}(\Omega) = x_j^0\left(x_i^0 - x_i^1\right) + x_j^1\left(x_i^1 - x_i^2\right) + \cdots + x_j^{p-1}\left(x_i^{p-1} - x_i^p\right). \tag{1}$$

We refer to the resulting matrix $C(\Omega) = (c_{ij}(\Omega))$ as the depathing map of Ω. (If Ω is a loop, then the depathing map $C(\Omega)$ is equivalent to the decirculating map defined in [7, 8], where we have explained why the terminology is used in connection with circulation breaking.) For example, let $\Omega = [1111100000, 0011111000, 0000111110, 0111110000, 0001111100]$. Then

$$C(\Omega) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -2 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the dynamical system of n coupled units modeled by the equation [6, 9]

$$x(t+1) = H_A\big(x(t), s(t)\big), \quad t = 0, 1, \ldots, \tag{2}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \{0,1\}^n$ is the vector of unit states at time t, $A = (a_{ij}) \in M_n(\mathbb{R})$ is the coupling matrix of n coupled units, $s(t) \subset \{1,2,\ldots,n\}$ denotes the units that adjust their states at time t, and $H_A(\cdot, s(t))$ is a function whose ith component is defined by

$$\left[H_A\big(x, s(t)\big)\right]_i = \mathbb{1}\left(\sum_{j=1}^n a_{ij}x_j - b_i\right) \quad \text{if } i \in s(t),$$

otherwise $[H_A(x, s(t))]_i = x_i$, where $b_i \in \mathbb{R}$ is the threshold of unit i and the function $\mathbb{1}$ is the Heaviside function: $\mathbb{1}(u) = 1$ for $u \geq 0$, otherwise 0, which describes an instantaneous unit pulse. The dynamical system generates the vector of unit states according to (2), resulting in the phase flow $x(t)$, $t = 0, 1, \ldots$.

With the depathing map $C(\Omega)$, we are bound to consider the linear functional $A \longrightarrow \langle A, C(\Omega)\rangle$ on the Hilbert space $M_n(\mathbb{R})$ of all real $n \times n$ matrices endowed with the Hilbert-Schmidt inner product $\langle\cdot,\cdot\rangle$.

Theorem 2.1 *Let $\Omega = [x^0, x^1, \ldots, x^p]$ be a path of states in $\{0,1\}^n$. If $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ satisfy*

$$\langle A, C(\Omega)\rangle \geq \langle b, x^0 - x^p\rangle, \tag{3}$$

then for any initial unit state $x(0) \in \{0,1\}^n$ and any updating $s(t) \subset \{1, 2, \ldots, n\}$, $t = 0, 1, \ldots$, the resulting phase flow $x(t)$ of (2) *cannot behave in*

$$x(T) = x^0, x(T+1) = x^1, \ldots, x(T+p) = x^p$$

for each $T = 0, 1, \ldots$.

Proof For any 01-string $x = x_1 x_2 \cdots x_n$, we define

$$\mathbf{1}(x) = \{i; x_i = 1, 1 \leq i \leq n\},$$
$$\mathbf{0}(x) = \{i; x_i = 0, 1 \leq i \leq n\}.$$

Suppose, by contradiction, that there exist $b \in \mathbb{R}^n$, $x(0) \in \{0,1\}^n$, $s(t) \subset \{1, 2, \ldots, n\}$, $t = 0, 1, \ldots$, and $T \geq 0$ such that $x(T) = x^0, x(T+1) = x^1, \ldots, x(T+p) = x^p$. Let

$$\Lambda^+ = \{t; \mathbf{0}(x(t)) \cap \mathbf{1}(x(t+1)) \neq \emptyset, T \leq t < T+p\},$$
$$\Lambda^- = \{t; \mathbf{1}(x(t)) \cap \mathbf{0}(x(t+1)) \neq \emptyset, T \leq t < T+p\}.$$

Then $\Lambda^+ \neq \emptyset$ and $\Lambda^- \neq \emptyset$. Indeed, if $\Lambda^+ = \emptyset$ or $\Lambda^- = \emptyset$, then

$$x(T) = x(T+1) = \cdots = x(T+p),$$

contradicting the path assumption $x(T) \neq x(T+p)$. According to (1), we have

$$\begin{aligned}
\langle A, C(\Omega)\rangle &= \sum_{i,j} a_{ij}\left(\sum_{0\leq m<p} x_j^m x_i^m - \sum_{0\leq m<p} x_j^m x_i^{m+1}\right) \\
&= \sum_{0\leq m<p}\left(\sum_{i,j} a_{ij}x_j^m x_i^m - \sum_{i,j} a_{ij}x_j^m x_i^{m+1}\right) \\
&= \sum_{0\leq m<p}\left(\langle Ax(T+m), x(T+m)\rangle - \langle Ax(T+m), x(T+m+1)\rangle\right) \\
&= \sum_{0\leq m<p}\langle Ax(T+m), x(T+m) - x(T+m+1)\rangle.
\end{aligned} \tag{4}$$

Since $\mathbf{0}(x(t)) \cap \mathbf{1}(x(t+1)) \subset s(t)$ and $\mathbf{1}(x(t)) \cap \mathbf{0}(x(t+1)) \subset s(t)$ for each $t = 0, 1, \ldots$, we conclude from (2) that

$$\begin{aligned}
&\sum_{0\le m<p} \bigl\langle Ax(T+m), x(T+m) - x(T+m+1)\bigr\rangle \\
&\quad < -\sum_{t\in\Lambda^+} \sum_{j\in\mathbf{0}(x(t))\cap\mathbf{1}(x(t+1))} b_j + \sum_{t\in\Lambda^-} \sum_{j\in\mathbf{1}(x(t))\cap\mathbf{0}(x(t+1))} b_j \\
&\quad = \sum_{0\le m<p} \bigl\langle b, x(T+m) - x(T+m+1)\bigr\rangle \\
&\quad = \bigl\langle b, x(T) - x(T+p)\bigr\rangle. \qquad (5)
\end{aligned}$$

Combining (4) and (5) shows that $\langle A, C(\Omega)\rangle < \langle b, x^0 - x^p\rangle$, contradicting (3), and that completes the proof. □

3 Operator control on path breaking

Denote by $\Omega = [x^0, x^1, \ldots, x^p]$ a path of states in $\{0,1\}^n$. For each $m = 0, 1, \ldots, p$, we say that the state x^m is in the position m of the path Ω. For each 01-string $x = x_1x_2\cdots x_n$, let

$$\begin{aligned}
&\mathbf{1}(x) = \{i; x_i = 1, 1 \le i \le n\}, \\
&\mathbf{0}(x) = \{i; x_i = 0, 1 \le i \le n\}.
\end{aligned}$$

Let us recall that the symmetric difference of two sets U and V is the set $U \triangle V$, each of whose elements belongs to U but not to V, or belongs to V but not to U. For every $i = 1, 2, \ldots, n$, let

$$\begin{aligned}
&M_i(\Omega) = \bigl\{m; i \in \mathbf{1}\bigl(x^{m-1}\bigr) \triangle \mathbf{1}\bigl(x^m\bigr), m = 1, 2, \ldots, p\bigr\}, \\
&M_i(\Omega)^+ = \bigl\{m; i \in \mathbf{1}\bigl(x^m\bigr) \setminus \mathbf{1}\bigl(x^{m-1}\bigr), m = 1, 2, \ldots, p\bigr\}, \qquad (6) \\
&M_i(\Omega)^- = \bigl\{m; i \in \mathbf{1}\bigl(x^{m-1}\bigr) \setminus \mathbf{1}\bigl(x^m\bigr), m = 1, 2, \ldots, p\bigr\}.
\end{aligned}$$

Here $M_i(\Omega)$ denotes the collection of the positions m of the path Ω, in which unit i changes its state from $x_i^{m-1} = 0$ to $x_i^m = 1$ or from $x_i^{m-1} = 1$ to $x_i^m = 0$, whereas $M_i(\Omega)^+$ (resp., $M_i(\Omega)^-$) denotes the collection of the positions m of the path Ω, in which unit i changes its state from $x_i^{m-1} = 0$ to $x_i^m = 1$ (resp., changes its state from $x_i^{m-1} = 1$ to $x_i^m = 0$). For every $i, j = 1, 2, \ldots, n$, define

$$\begin{aligned}
\Upsilon_{ij}(\Omega) = {}& \sharp\bigl(M_i(\Omega)^+ \cap M_j(\Omega)^+\bigr) + \sharp\bigl(M_i(\Omega)^- \cap M_j(\Omega)^-\bigr) \\
& - \sharp\bigl(M_i(\Omega)^+ \cap M_j(\Omega)^-\bigr) - \sharp\bigl(M_i(\Omega)^- \cap M_j(\Omega)^+\bigr), \qquad (7)
\end{aligned}$$

which can be regarded as a measure of synchronous activity between units i, j, that is, if units i, j tend to change their states synchronously (resp., asynchronously) in Ω, then $\Upsilon_{ij}(\Omega) > 0$ (resp., $\Upsilon_{ij}(\Omega) < 0$). For every $i, j = 1, 2, \ldots, n$, define

$$\Gamma_{ij}(\Omega) = \min\bigl\{\sharp M_i(\Omega), \sharp M_j(\Omega)\bigr\}, \qquad (8)$$

which can be regarded as a measure of self-sustaining activity of units i,j, that is, if unit i or j tends to maintain more self-sustaining states in Ω than unit i' or j', then $\Gamma_{ij}(\Omega) < \Gamma_{i'j'}(\Omega)$. We refer to the resulting matrices $\Upsilon(\Omega) = (\Upsilon_{ij}(\Omega))$ and $\Gamma(\Omega) = (\Gamma_{ij}(\Omega))$ as the measure of synchronous activity and the measure of self-sustaining activity derived from the path Ω of states in $\{0,1\}^n$, respectively.

Let $\overline{\Omega} = [x^0, x^1, \ldots, x^p, x^0]$. Denote by $\langle\cdot,\cdot\rangle$ the Hilbert-Schmidt inner product in $M_n(\mathbb{R})$, *i.e.*, if $A = (a_{ij})$ and $B = (b_{ij}) \in M_n(\mathbb{R})$, then $\langle A, B\rangle = \operatorname{tr}(AB^T) = \sum_{i,j} a_{ij}b_{ij}$. Define $I(\Omega) = \mathbf{1}(x^p) \cap \mathbf{1}(x^0)$. Let

$$\begin{aligned}\mathcal{D}(\Omega) = \bigl\{&\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}; \mathcal{D}_{\mathrm{SY}} = \mathcal{D}_{\mathrm{SY}}^T \in M_n(\mathbb{R}), \mathcal{D}_{\mathrm{SK}} = -\mathcal{D}_{\mathrm{SK}}^T \in M_n(\mathbb{R}) \text{ with} \\ &(\mathcal{D}_{\mathrm{SK}})_{ij} \geq 0 \text{ for each } (i,j) \in \bigl(\mathbf{1}(x^0) \times \mathbf{1}(x^p)\bigr) \setminus \bigl(I(\Omega) \times I(\Omega)\bigr), \text{ and} \\ &\bigl\langle |\mathcal{D}_{\mathrm{SK}}|, \Gamma(\overline{\Omega})\bigr\rangle < \bigl\langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega}) - C\bigl([x^p, x^0]\bigr)\bigr\rangle\bigr\},\end{aligned} \tag{9}$$

where $|\mathcal{D}_{\mathrm{SK}}| = (|(\mathcal{D}_{\mathrm{SK}})_{ij}|)$. The set $\mathcal{D}(\Omega)$ collects all the combining operations of the operators $\mathcal{D}_{\mathrm{SY}}$ and $\mathcal{D}_{\mathrm{SK}}$, which will determine a clamp of network modification by $A + \mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}$.

Fix $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}} \in \mathcal{D}(\Omega)$. Define $I_1(\Omega) = \mathbf{1}(x^p) \cap \mathbf{0}(x^0)$ and $I_2(\Omega) = \mathbf{0}(x^p) \cap \mathbf{1}(x^0)$. Since $I_1(\Omega) \cap I(\Omega) = \emptyset$, $I_1(\Omega) \cup I(\Omega) = \mathbf{1}(x^p)$, $I_2(\Omega) \cap I(\Omega) = \emptyset$, and $I_2(\Omega) \cup I(\Omega) = \mathbf{1}(x^0)$, we have

$$\bigl(\mathbf{1}(x^0) \times \mathbf{1}(x^p)\bigr) \setminus \bigl(I(\Omega) \times I(\Omega)\bigr) = \bigl(I(\Omega) \times I_1(\Omega)\bigr) \cup \bigl(I_2(\Omega) \times \mathbf{1}(x(p))\bigr),$$

and hence, by (9),

$$\begin{aligned}\bigl\langle \mathcal{D}_{\mathrm{SK}}, C\bigl([x^p, x^0]\bigr)\bigr\rangle &= \sum_{(i,j)\in I_1(\Omega)\times\mathbf{1}(x(p))} (\mathcal{D}_{\mathrm{SK}})_{ij} - \sum_{(i,j)\in I_2(\Omega)\times\mathbf{1}(x(p))} (\mathcal{D}_{\mathrm{SK}})_{ij} \\ &= \sum_{(i,j)\in I_1(\Omega)\times I_1(\Omega)} (\mathcal{D}_{\mathrm{SK}})_{ij} + \sum_{(i,j)\in I_1(\Omega)\times I(\Omega)} (\mathcal{D}_{\mathrm{SK}})_{ij} - \sum_{(i,j)\in I_2(\Omega)\times\mathbf{1}(x(p))} (\mathcal{D}_{\mathrm{SK}})_{ij} \\ &\leq 0.\end{aligned} \tag{10}$$

Furthermore, according to the proof in [8, Theorem 1], the following assertion holds:

$$\bigl\langle \mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}, C(\overline{\Omega})\bigr\rangle \geq \bigl\langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega})\bigr\rangle - \bigl\langle |\mathcal{D}_{\mathrm{SK}}|, \Gamma(\overline{\Omega})\bigr\rangle. \tag{11}$$

Combining (9), (10), and (11) shows that

$$\begin{aligned}\bigl\langle \mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}, C(\Omega)\bigr\rangle &= \bigl\langle \mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}, C(\overline{\Omega}) - C\bigl([x^p, x^0]\bigr)\bigr\rangle \\ &\geq \bigl\langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega})\bigr\rangle - \bigl\langle |\mathcal{D}_{\mathrm{SK}}|, \Gamma(\overline{\Omega})\bigr\rangle \\ &\quad - \bigl\langle \mathcal{D}_{\mathrm{SY}}, C\bigl([x^p, x^0]\bigr)\bigr\rangle - \bigl\langle \mathcal{D}_{\mathrm{SK}}, C\bigl([x^p, x^0]\bigr)\bigr\rangle \\ &> 0.\end{aligned}$$

With the notation and arguments above, we describe operator control on path breaking as follows.

Theorem 3.1 *Let $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Let $\Omega = [x^0, x^1, \ldots, x^p]$ be a path of states in $\{0,1\}^n$. Then, for every operator $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}} \in \mathcal{D}(\Omega)$, there exists $\gamma \geq 0$ such that*

$$\bigl\langle A + \gamma(\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}), C(\Omega)\bigr\rangle \geq \bigl\langle b, x^0 - x^p\bigr\rangle.$$

Hence, by Theorem 2.1, *the dynamical system of* n *coupled units modeled by the equation*

$$x(t+1) = H_{A+\gamma(\mathcal{D}_{\mathrm{SY}}+\mathcal{D}_{\mathrm{SK}})}\bigl(x(t), s(t)\bigr), \quad t = 0, 1, \ldots,$$

cannot behave in

$$x(T) = x^0, x(T+1) = x^1, \ldots, x(T+p) = x^p$$

for each $T = 0, 1, \ldots$.

In the following, we give an example to construct a sort of operators $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}} \in \mathcal{D}(\Omega)$. Let $n = 10$ and $p = 4$. Let $\Omega = [x^0, x^1, x^2, x^3, x^4] = [1111100000, 0011111000, 0000111110, 0111110000, 0001111100]$. Then

$$\begin{aligned}
I(\Omega) &= \mathbf{1}\bigl(x^p\bigr) \cap \mathbf{1}\bigl(x^0\bigr) = \{4, 5\}, \\
I_1(\Omega) &= \mathbf{1}\bigl(x^p\bigr) \cap \mathbf{0}\bigl(x^0\bigr) = \{6, 7, 8\}, \\
I_2(\Omega) &= \mathbf{0}\bigl(x^p\bigr) \cap \mathbf{1}\bigl(x^0\bigr) = \{1, 2, 3\}.
\end{aligned}$$

Associate to each $i \in \{1, 2, \ldots, n\}$ a real number ε_i such that

(1) for each $i, j \in \{1, 2, \ldots, n\}$, if $(\sum_m x_i^m)/(M_i(\Omega)+1) > (\sum_m x_j^m)/(M_j(\Omega)+1)$, then $\varepsilon_i \le \varepsilon_j$;

(2) $\sum_{1\le i\le n} M_i(\Omega)\varepsilon_i + \sum_{i\in I_2(\Omega)} \varepsilon_i > 0$

(see Table 1 for a choice of ε_i). We may select $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\mathcal{D}_{\mathrm{SY}} = (y_i y_j + \delta_{ij}\varepsilon_i) \in M_n(\mathbb{R})$, where $\delta_{ij} = 1$ if $i = j$, otherwise 0, such that exactly one of the following holds:

(1) $y_i \ge 0$ for $i \in \mathbf{1}(x^p)$ and $\sum_{i\in I_1(\Omega)} y_i \le \sum_{i\in I_2(\Omega)} y_i$;

(2) $y_i \le 0$ for $i \in \mathbf{1}(x^p)$ and $\sum_{i\in I_1(\Omega)} y_i \ge \sum_{i\in I_2(\Omega)} y_i$

(see Table 1 for a choice of y_i and Table 2 for $\mathcal{D}_{\mathrm{SY}}$). Consider the shift function σ on $\{0, 1, \ldots, p\}$ given by

$$\sigma(k) \equiv k+1 \mod p+1$$

for each $k = 0, 1, \ldots, p$. Since $(y_i y_j) \in M_n(\mathbb{R})$ is positive semidefinite, we have

$$\begin{aligned}
\bigl\langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega}) - C\bigl(\bigl[x^p, x^0\bigr]\bigr)\bigr\rangle &= \sum_{0\le m\le p} \bigl\langle \mathcal{D}_{\mathrm{SY}}\bigl(x^m - x^{\sigma(m)}\bigr), \bigl(x^m - x^{\sigma(m)}\bigr)\bigr\rangle \\
&\quad - \sum_{j\in\mathbf{1}(x^p)} y_j\Biggl(\sum_{i\in I_1(\Omega)} y_i - \sum_{i\in I_2(\Omega)} y_i\Biggr) - \sum_{i\in I_1(\Omega)} \varepsilon_i \\
&\ge \sum_{0\le m<p} \sum_{i\in\mathbf{1}(x^m)\,\triangle\,\mathbf{1}(x^{\sigma(m)})} \varepsilon_i + \sum_{i\in I_2(\Omega)} \varepsilon_i \\
&= \sum_{1\le i\le n} M_i(\Omega)\varepsilon_i + \sum_{i\in I_2(\Omega)} \varepsilon_i > 0.
\end{aligned}$$

For such a choice of $\mathcal{D}_{\mathrm{SY}}$, let $\varepsilon > 0$ be such that

$$\gamma_\Omega = \bigl(\bigl\langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega}) - C\bigl(\bigl[x^p, x^0\bigr]\bigr)\bigr\rangle - \varepsilon\bigr)/\bigl\|\Gamma(\overline{\Omega})\bigr\| \ge 0, \tag{12}$$

Table 1 Operator construction: choose ε_i and y_i

$\Omega = [1111100000, 0011111000, 0000111110, 0111110000, 0001111100]$										
i	1	2	3	4	5	6	7	8	9	10
$M_i(\Omega)$	1	3	3	2	0	1	3	3	2	0
$\sum_m x_i^m$	1	2	3	4	5	4	3	2	1	0
$(\sum_m x_i^m)/(M_i(\Omega)+1)$	1/2	1/2	3/4	4/3	5	2	3/4	1/2	1/3	0
ε_i	0.8	0.5	0.5	−0.7	−1.8	−1	−0.3	0.5	1.2	1.5
y_i	3	1	−1	0	0.5	1	1	1	0	−1

Table 2 Operator construction: construct the operator $\mathcal{D}_{\mathrm{SY}} = (y_i y_j + \delta_{ij}\varepsilon_i)$

9.8	3	−3	0	1.5	3	3	3	0	−3
3	1.5	−1	0	0.5	1	1	1	0	−1
−3	−1	1.5	0	−0.5	−1	−1	−1	0	1
0	0	0	−0.7	0	0	0	0	0	0
1.5	0.5	−0.5	0	−1.55	0.5	0.5	0.5	0	−0.5
3	1	−1	0	0.5	0	1	1	0	−1
3	1	−1	0	0.5	1	0.7	1	0	−1
3	1	−1	0	0.5	1	1	1.5	0	−1
0	0	0	0	0	0	0	0	1.2	0
−3	−1	1	0	−0.5	−1	−1	−1	0	2.5

Table 3 Operator construction: construct the operator $\mathcal{D}_{\mathrm{SK}} = \alpha S$ satisfying (12) and (13)

0	0	−0.015	0	0.005	0	0.02	0.01	−0.015	−0.01
0	0	0.01	0	0.015	0.01	0.015	0.015	0.01	−0.005
0.015	−0.01	0	0.015	0	0.01	0.01	0.01	0.005	0.01
0	0	−0.015	0	−0.005	0.005	0.01	0.01	0	−0.01
−0.005	−0.015	0	0.005	0	0.005	0.01	0.005	0.015	0
0	−0.01	−0.01	−0.005	−0.005	0	−0.01	0.015	0.01	−0.02
−0.02	−0.015	−0.01	−0.01	−0.01	0.01	0	0.005	0.015	0.005
−0.01	−0.015	−0.01	−0.01	−0.005	−0.015	−0.005	0	0.005	0.015
0.015	−0.01	−0.005	0	−0.015	−0.01	−0.015	−0.005	0	0
0.01	0.005	−0.01	0.01	0	0.02	−0.005	−0.015	0	0

Table 4 Operator construction: construct the combining operation $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}$

9.8	3	−3.015	0	1.505	3	3.02	3.01	−0.015	−3.01
3	1.5	−0.99	0	0.515	1.01	1.015	1.015	0.01	−1.005
−2.985	−1.01	1.5	0.015	−0.5	−0.99	−0.99	−0.99	0.005	1.01
0	0	−0.015	−0.7	−0.005	0.005	0.01	0.01	0	−0.01
1.495	0.485	−0.5	0.005	−1.55	0.505	0.51	0.505	0.015	−0.5
3	0.99	−1.01	−0.005	0.495	0	0.99	1.015	0.01	−1.02
2.98	0.985	−1.01	−0.01	0.49	1.01	0.7	1.005	0.015	−0.995
2.99	0.985	−1.01	−0.01	0.495	0.985	0.995	1.5	0.005	−0.985
0.015	−0.01	−0.005	0	−0.015	−0.01	−0.015	−0.005	1.2	0
−2.99	−0.995	0.99	0.01	−0.5	−0.98	−1.005	−1.015	0	2.5

where $\|\Gamma(\overline{\Omega})\| = \langle \Gamma(\overline{\Omega}), \Gamma(\overline{\Omega})\rangle^{\frac{1}{2}}$. Then, for any choice of $S \in M_n(\mathbb{R})$ with $S = -S^T$ and $S_{ij} \geq 0$ for each $(i,j) \in (\mathbf{1}(x^0) \times \mathbf{1}(x^p)) \setminus (I(\Omega) \times I(\Omega))$, we set $\mathcal{D}_{\mathrm{SK}} = \alpha S$, where $\alpha \in \mathbb{R}$ is such that

$$\|\mathcal{D}_{\mathrm{SK}}\| = \langle \mathcal{D}_{\mathrm{SK}}, \mathcal{D}_{\mathrm{SK}}\rangle^{\frac{1}{2}} = \alpha \langle S, S\rangle^{\frac{1}{2}} \leq \gamma_\Omega \tag{13}$$

(see Table 3 for a choice of $\mathcal{D}_{\mathrm{SK}}$). Thus, by (12) and (13), we have

$$\langle |\mathcal{D}_{\mathrm{SK}}|, \Gamma(\overline{\Omega})\rangle < \langle \mathcal{D}_{\mathrm{SY}}, \Upsilon(\overline{\Omega}) - C([x^p, x^0])\rangle.$$

Hence $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}} \in \mathcal{D}(\Omega)$ (see Table 4 for a choice of $\mathcal{D}_{\mathrm{SY}} + \mathcal{D}_{\mathrm{SK}}$).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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